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\[
\sum_{n=0}^{\infty} \frac{16n^2 + 4n - 1}{(4n + 2)!}
\]
\[ \sum_{n=0}^{\infty} \frac{16n^2 + 4n - 1}{(4n + 2)!} = \frac{16(0)^2 + 4(0) - 1}{(4(0) + 2)!} + \frac{16(1)^2 + 4(1) - 1}{(4(1) + 2)!} + \frac{16(2)^2 + 4(2) - 1}{(4(2) + 2)!} + \cdots \]
Maclaurin Series:

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

\[ \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \]
Euler’s Identity:

\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \]

Proof:

\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \frac{1}{0!} + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots
\]

\[
= \left( \frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \right) + i \left( \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \right)
\]

\[
= \cos(\theta) + i \sin(\theta)
\]

\[\square\]
4th roots of unity:

\[ z^4 = 1 \] has solutions 1, -1, i, and -i since \( (1)^4 = 1, (-1)^4 = 1, (i)^4 = 1, (-i)^4 = 1 \)
\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, z = 1, -1, i, -i
\]

\[
\frac{e^1 + e^{-1} - (e^i + e^{-i})}{4} = \left( \sum_{n=0}^{\infty} \frac{1^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - \sum_{n=0}^{\infty} \frac{i^n}{n!} - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \right)
\]

\[
= \left[ \left( \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!} + \cdots \right) \right]
\]

\[
+ \left[ \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!} + \cdots \right) \right]
\]

\[
- \left[ \left( \frac{1}{0!} + \frac{i}{1!} - \frac{1}{2!} + \frac{i}{3!} + \frac{1}{4!} - \frac{i}{5!} - \frac{1}{6!} + \frac{i}{7!} + \frac{1}{8!} + \frac{i}{9!} - \frac{1}{10!} + \cdots \right) \right]
\]

\[
- \left[ \left( \frac{1}{0!} - \frac{i}{1!} + \frac{1}{2!} - \frac{i}{3!} - \frac{1}{4!} + \frac{i}{5!} + \frac{1}{6!} - \frac{i}{7!} - \frac{1}{8!} + \frac{i}{9!} - \frac{1}{10!} + \cdots \right) \right]
\]
\[ \sum_{n=0}^{\infty} \frac{n z^n}{n!} = \sum_{n=1}^{\infty} \frac{n z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!} = z \sum_{n=0}^{\infty} \frac{z^n}{n!} = z e^z \]

\[ z = 1, z = -1, z = i, z = -i \]

\[ \frac{(e - e^{-1}) - (ie^i - ie^{-i})}{4} \]

\[ = \left[ 0 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \frac{5}{5!} + \frac{6}{6!} + \frac{7}{7!} + \cdots + 0 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \frac{4}{4!} - \frac{5}{5!} + \frac{6}{6!} - \frac{7}{7!} + \cdots - \right] \]
\[
\sum_{n=0}^{\infty} \frac{n^2 z^n}{n!} = \sum_{n=1}^{\infty} \frac{n^2 z^n}{n(n-1)!} = \sum_{n=0}^{\infty} \frac{(n+1)z^{n+1}}{n} = \sum_{n=0}^{\infty} \frac{n z^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}
\]

\[
= \sum_{n=1}^{\infty} \frac{z^{n+1}}{(n-1)!} + z e^z = \sum_{n=0}^{\infty} \frac{z^{n+2}}{n!} + z e^z = z^2 \sum_{n=0}^{\infty} \frac{z^n}{n!} + z e^z = z^2 e^z + ze^z
\]

\[
\frac{(2e + 0) - (-e^i + ie^i - e^{-i} - ie^{-i})}{4}
\]

\[
\left[ \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \frac{5^2}{5!} + \frac{6^2}{6!} + \frac{7^2}{7!} + \cdots - \frac{1}{1!} + \frac{2^2}{2!} - \frac{3^2}{3!} + \frac{4^2}{4!} - \frac{5^2}{5!} + \frac{6^2}{6!} - \frac{7^2}{7!} + \cdots \right] - \left( \frac{i}{1!} - \frac{2^2}{2!} - \right)
\]
\[ \sum_{n=0}^{\infty} \frac{(4n + 2)^2}{(4n + 2)!} \]

\[ \sum_{n=0}^{\infty} \frac{4n + 2}{(4n + 2)!} \]

\[ \sum_{n=0}^{\infty} \frac{1}{(4n + 2)!} \]
\[
\sum_{n=0}^{\infty} \frac{(4n+2)^2 - 3(4n+2) + 1}{(4n+2)!} = \sum_{n=0}^{\infty} \frac{(4n+2)^2}{(4n+2)!} - 3 \sum_{n=0}^{\infty} \frac{4n+2}{(4n+2)!} + \sum_{n=0}^{\infty} \frac{1}{(4n+2)!} \\
= \frac{2e + e^i - ie^i + e^{-i} + ie^{-i}}{4} - 3 \left( \frac{e - e^{-1} - ie^i + ie^{-i}}{4} \right) + \frac{e + e^{-1} - e^i - e^{-i}}{4} \\
= \frac{2e + e^i - ie^i + e^{-i} + ie^{-i} - 3e + 3e^{-1} + 3ie - 3ie^{-i} + e + e^{-1} - e^i - e^{-i}}{4}
\]
\[ = 4e^{-1} + 2ie^i - 2ie^{-i} \]

Euler’s Identity

\[ = \frac{4e^{-1} + 2i[\cos(1) + i \sin(1) - \cos(-1) - i \sin(-1)]}{4} \]

\[ = \frac{4e^{-1} + 4i^2 \sin(1)}{4} \]

\[ = e^{-1} - \sin(1) \]